

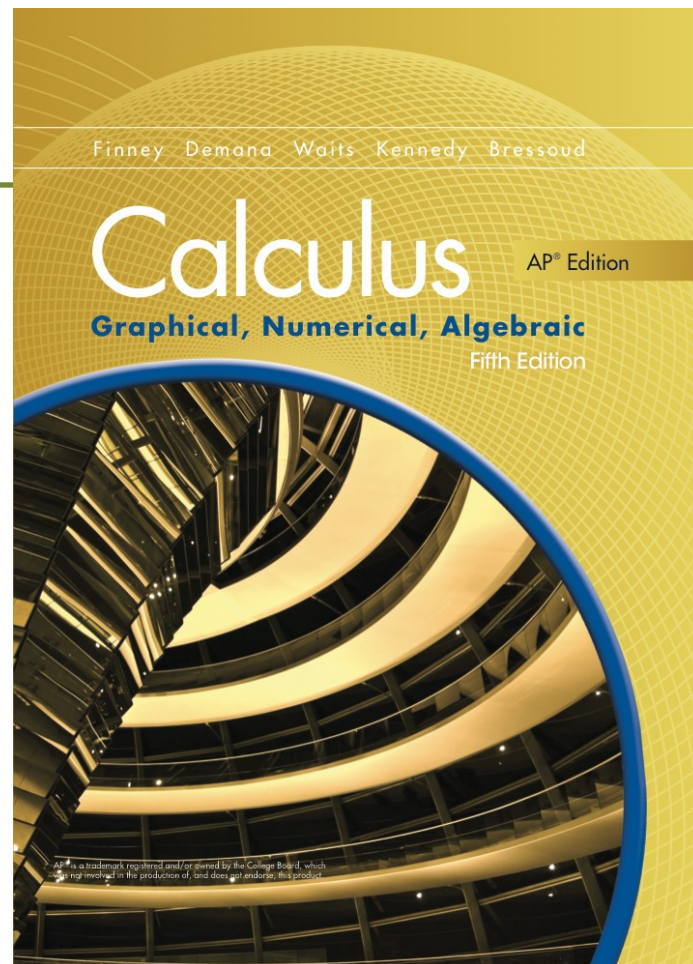
Chapter 9

Sequences, L'Hopital's Rule, and Improper Integrals



Section 9.3

Relative Rates of Growth



Quick Review

Evaluate the limit

1. $\lim_{x \rightarrow \infty} \frac{\ln x}{e^x}$

2. $\lim_{x \rightarrow \infty} \frac{e^x}{x^3}$

3. $\lim_{x \rightarrow -\infty} \frac{x^2}{e^{2x}}$

4. $\lim_{x \rightarrow \infty} \frac{x^2}{e^{2x}}$

Find the end behavior model for the function

5. $f(x) = -3x^4 + 3x^2 + x - 1$

6. $f(x) = \frac{2x^3 + 4x + 2}{x - 3}$

Quick Review

7. Let $f(x) = \frac{e^x + x^2}{e^x}$. Find the

- (a) local extreme values of f and where they occur.
- (b) intervals on which f is increasing.
- (c) intervals on which f is decreasing.

Quick Review Solutions

Evaluate the limit

$$1. \lim_{x \rightarrow \infty} \frac{\ln x}{e^x} \quad 0$$

$$2. \lim_{x \rightarrow \infty} \frac{e^x}{x^3} \quad \infty$$

$$3. \lim_{x \rightarrow -\infty} \frac{x^2}{e^{2x}} \quad \infty$$

$$4. \lim_{x \rightarrow \infty} \frac{x^2}{e^{2x}} \quad 0$$

Find the end behavior model for the function

$$5. f(x) = -3x^4 + 3x^2 + x - 1 \quad -3x^4$$

$$6. f(x) = \frac{2x^3 + 4x + 2}{x - 3} \quad 2x^2$$

Quick Review Solutions

7. Let $f(x) = \frac{e^x + x^2}{e^x}$. Find the

(a) local extreme values of f and where they occur.

Local minimum at $(0,1)$. Local maximum at $\left(2, \frac{e^2 + 4}{e^2}\right)$.

(b) intervals on which f is increasing. $[0,2]$

(c) intervals on which f is decreasing. $(-\infty, 0]$ and $[2, \infty)$

What you'll learn about

- Comparing growth rates
- Comparing growth rates using L'Hôpital's Rule
- Relative efficiency of sequential and binary search algorithms

...and why

Understanding growth rates as $x \rightarrow \infty$ is an important feature in understanding the behavior of functions.

Faster, Slower, Same-rate Growth as $x \rightarrow \infty$

Let $f(x)$ and $g(x)$ be positive for x sufficiently large.

1. f **grows faster** than g (and g **grows slower** than f) as $x \rightarrow \infty$ if

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \infty, \text{ or, equivalently, if } \lim_{x \rightarrow \infty} \frac{g(x)}{f(x)} = 0.$$

2. f and g **grow at the same rate** as $x \rightarrow \infty$ if

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = L \neq 0.$$

Example Comparing e^x and x^3 as $x \rightarrow \infty$

Show that the function e^x grows faster than x^3 as $x \rightarrow \infty$.

Show that $\lim_{x \rightarrow \infty} \frac{e^x}{x^3} = \infty$. Notice that the limit is of the indeterminate form ∞/∞ , so we can apply l'Hôpital's Rule.

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^3} = \lim_{x \rightarrow \infty} \frac{e^x}{3x^2} = \lim_{x \rightarrow \infty} \frac{e^x}{6x} = \lim_{x \rightarrow \infty} \frac{e^x}{6} = \infty$$

Example Comparing $\ln x$ with x as $x \rightarrow \infty$

Show that $\ln x$ grows slower than x .

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{1/x}{1} = \lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

Example Comparing x with $x + \sin x$ as $x \rightarrow \infty$

Show that x grows at the same rate as $x + \sin x$ as $x \rightarrow \infty$.

Show that $\lim_{x \rightarrow \infty} \frac{x + \sin x}{x}$ is finite and not 0.

$$\lim_{x \rightarrow \infty} \frac{x + \sin x}{x} = \lim_{x \rightarrow \infty} \left(1 + \frac{\sin x}{x} \right) = 1$$

Transitivity of Growing Rates

If f grows at the same rate as g as $x \rightarrow \infty$ and g grows at the same rate as h as $x \rightarrow \infty$, then f grows at the same rate as h as $x \rightarrow \infty$.

Example Growing at the Same Rate

as $x \rightarrow \infty$

Show that $f(x) = \sqrt{x^2 + 4}$ and $g(x) = (2\sqrt{x} - 1)^2$ grow at the same rate as $x \rightarrow \infty$.

Show that f and g grow at the same rate by showing that they both grow at the same rate as $h(x) = x$

$$\lim_{x \rightarrow \infty} \frac{f(x)}{h(x)} = \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 4}}{x} = \lim_{x \rightarrow \infty} \sqrt{1 + \frac{4}{x^2}} = 1 \quad \text{and}$$

$$\lim_{x \rightarrow \infty} \frac{g(x)}{h(x)} = \lim_{x \rightarrow \infty} \frac{(2\sqrt{x} - 1)^2}{x} = \lim_{x \rightarrow \infty} \left(\frac{2\sqrt{x} - 1}{\sqrt{x}} \right)^2 = \lim_{x \rightarrow \infty} \left(2 - \frac{1}{\sqrt{x}} \right)^2 = 4$$

$$\text{Thus, } \lim_{x \rightarrow \infty} \frac{fg}{g} = \lim_{x \rightarrow \infty} \left(\frac{f}{h} \cdot \frac{h}{g} \right) = 1 \cdot \frac{1}{4} = \frac{1}{4}.$$

Example Finding the Order of a Binary Search

For a list of length n , how many steps are required for a binary search?

A binary search takes on the order of $\log_2 n$ steps. The reason is if $2^{m-1} < n \leq 2^m$, then $m-1 < \log_2 n \leq m$ and the number of bisections required to narrow the list to one word will be at most m , the smallest integer greater or equal to $\log_2 n$.

Sequential Versus Binary Search

One way to look up a word, or to learn if it is not there, is to read through the list one word at a time until you either find the word or determine that it is not there. This **sequential search** method makes no particular use of the words' alphabetical arrangement. You are sure to get an answer, but it might take about 26,000 steps.

Sequential Versus Binary Search

Another way to find the word or to learn that it is not there is to go straight to the middle of the list (give or take a few words). If you do not find the word, then go to the middle of the half that would contain it and forget about the half that would not. This **binary search** method eliminates roughly 13,000 words in this first step.

Sequential Versus Binary Search

For a list of length n , a **sequential search** algorithm takes on the order of **n steps** to find a word or determine that it is not in the list.

For a list of length n , a **binary search** takes on the order of **$\log_2 n$ steps**.